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On the Relationship Between Natural Frequency and -3dB Bandwidth for a Second-Order System

Second-order, negative feedback systems have both a –3dB (or, half-power) bandwidth and a natural frequency of oscillation. When choosing design characteristics for such systems, it can be useful to know how these parameters are related to each other.

The transfer function for a second-order system is:

$$\mathrm{H}(s) = \frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}} \tag{1}$$

where: ω_n is the natural frequency of the loop, in radians/s
ζ is the loop damping factor, between 0 and 1

Substituting for the frequency domain, we get:

$$\mathrm{h}(j\omega) = \frac{\omega_{\mathrm{n}}^{\;2}}{(j\omega)^2 + 2\zeta\omega_n (j\omega) + \omega_n^{\;2}} \tag{2}$$

This can be re-written as:

$$\mathrm{h}(j\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)} \tag{3}$$

To derive the system bandwidth, we first want transfer function magnitude:

$$\left|\mathrm{h}(j\omega)\right| = \frac{1}{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2\right]^{\frac{1}{2}}} \tag{4}$$

Bandwidth is defined as the point where output power falls to half its low-frequency value:

$$\left|\mathrm{h}(j\omega)\right|^2 = \frac{1}{2} \tag{5}$$

Combining (4) and (5), taking the inverse, and collecting terms to one side, we get:

$$\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2 - 2 = 0 \tag{6}$$

If we expand, and re-combine terms, we get:

$$\left(\frac{\omega}{\omega_n}\right)^4 + 2\left(\frac{\omega}{\omega_n}\right)^2 \left(2\zeta^2 - 1\right) - 1 = 0 \qquad (7)$$

This expression is quadratic in $\left(\frac{\omega}{\omega_n}\right)^2$. If we solve the quadratic for the positive root, we get:

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(1 - 2\zeta^2\right) + \sqrt{\left(2\zeta^2 - 1\right)^2 + 1} \qquad (8)$$

Now, the 3dB bandwidth is:

$$\omega_{3\mathrm{dB}} = \omega_n \sqrt{\left(1 - 2\zeta^2\right) + \sqrt{\left(2\zeta^2 - 1\right)^2 + 1}} \qquad (9)$$

At this point, it's interesting to note that expression (9) is not reported in precisely the same way everywhere. In particular, one version of the PCI-E Base Specification shows plus signs where the above expression shows minus signs[1]. This indicates either that there is a typo in that document, or that there is an error in the analysis. To verify this, suppose we try a test case.

A simple test case:

When is Natural Frequency the same as -3dB Bandwidth?

From (9), normalizing and squaring:

$$1 = \left(1 - 2\zeta^2\right) + \sqrt{\left(2\zeta^2 - 1\right)^2 + 1} \qquad (T1)$$

Re-arranging terms:

$$2\zeta^2 = \sqrt{\left(2\zeta^2 - 1\right)^2 + 1} \qquad (T2)$$

Squaring again, and expanding:

$$4\zeta^4 = 4\zeta^4 - 4\zeta^2 + 1 + 1 \qquad (T3)$$

Re-arranging terms:

$$\zeta^2 = \frac{1}{2},\ \zeta = \frac{1}{\sqrt{2}} = 0.7071 \ \ \text{(Critical Damping)} \qquad (T4)$$

Notice that the result, (T4), is a 'real' value. When plus signs are substituted in (9), we get an imaginary value for damping factor. A system with an imaginary damping factor is not realizable.

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[1] Expression 4-5, page 227 of "PCI Express Base Specification Rev 1.1, March 28 2005", http://www.pcisig.com/home